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On a Uniqueness Theorem of Privalov*

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Let $f(z)$ be a meromorphic function in the open unit disk U with values on the Riemann sphere Ω. Denote by $D(f)$ the set of points ζ on the unit circle K at which the radial limit (finite or infinite),

$$\lim_{r\to 1} f(r\zeta) = \Gamma_\rho(f, \zeta),$$

exists, and by $\Gamma_\rho(f)$ the set of all radial limits of f. If A is a subarc of K, put

$$\bigcup_{\zeta\in A\cap D(f)} \Gamma_\rho(f, \zeta) = \Gamma_\rho(f, A).$$

The following is a well-known uniqueness theorem due to Privalov (see [1], p. 231).

THEOREM P. *Let $f(z)$ be meromorphic in U. Suppose that $D(f) \cap A$ is a metrically dense subset of A of second category. Then either $\Gamma_\rho(f)$ contains a closed set of positive harmonic measure, or f is identically constant.*

An improvement of this theorem was recently obtained by Cartwright and Collingwood [2, p. 404, Theorem 1], who weakened the hypothesis and strengthened the conclusion in the following manner.

THEOREM C_1. *Let $f(z)$ be meromorphic in U. Suppose that $D(f) \cap A$ is of second category. Then either $\Gamma_\rho(f, A)$ is of positive linear measure, or f is identically constant.*

In fact, the same authors proved an even more general result [2, p. 406, Theorem 2]. Define $E(f)$ to be the set of points $\zeta \in K$ for which $C_\rho(f, \zeta)$, the radial cluster set of f at ζ, is a proper subset of Ω.

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THEOREM C_2. *Let $f(z)$ be meromorphic in U. Suppose that $E(f) \cap A$ is of second category. Then either $\Gamma_\rho(f, A)$ is of positive linear measure, or f is identically constant.*

The general purpose of this note is to investigate the extent to which the approach to ζ in the assumptions of these theorems is necessarily radial or along curves that are rotational transforms of one another.

By an arc at $\zeta \in K$ we mean a simple continuous curve

$$\Lambda : z = z(t)\,(0 \leqslant t < 1)$$

such that $|z(t)| < 1$ for $0 \leqslant t < 1$ and $\lim_{t\to 1} z(t) = \zeta$. In particular, if Λ is rectilinear, we speak of a chord at ζ.

Now it has been shown [3, p. 79, Theorem 1] that there exists a function $f(z)$, holomorphic and not identically constant in U, with the property that for every point $\zeta \in K, f(z) \to \infty$ along a chord at ζ, but $\Gamma_\rho(f) \subseteqq \{\infty\}$, so that $\Gamma_\rho(f)$ is of linear measure zero. This shows that in Theorem C_1 the hypothesis concerning radial limits cannot be replaced by an analogous one involving approach to ζ along otherwise unspecified chords.

It is conceivable that ∞ plays a decisive role in the foregoing example because of the fact that a holomorphic function omits the value ∞. That this is actually not so will be shown after the proof of the following result.

THEOREM 1. *There exists a function $f(z)$, holomorphic and not identically constant in U, and a set X of chords with the property that at every point $\zeta \in K$ there is a chord belonging to X, such that $f(z) \to 0$ uniformly as z tends to K along the chords of X.*

PROOF. Consider the set S of all points on the circle $|z| = \frac{1}{2}$ of the form

$$z = \frac{1}{2}\, e^{\pm i(0 \cdot t_1 t_2 t_3 \ldots)\pi},$$

where $0 \cdot t_1 t_2 t_3 \ldots$ is a ternary fraction in which each t_j is either 0 or 2. For every $z \in S$, let χ_z be the chord extending from the point z to the point

$$\zeta = e^{\pm i(0 \cdot b_1 b_2 b_3 \ldots)\pi} \in K,$$

where $0 \cdot b_1 b_2 b_3 \ldots$ is the binary fraction such that, for $j = 1, 2, 3, \ldots$,

$$b_j = \begin{cases} 0 & \text{if } \; t_j = 0, \\ 1 & \text{if } \; t_j = 2. \end{cases}$$

If we join each χ_z to the origin by means of a rectilinear segment, we obtain a tress to which [4, p. 190, Corollary 2 (with $\alpha = \beta = 0$)] applies, and Theorem 1 follows immediately.

Now let $f(z)$ be any function possessing the properties described in Theorem 1. Then it follows from my ambiguous-point theorem [5, p. 382, Corollary 1] that the set $\Gamma_\rho(f)$ is at most enumerable, and hence is of linear measure zero.

Although it would appear then that the radial approach (or approach along arcs that are rotational transforms of one another) is essential in the Privalov-type theorem, we shall show nevertheless that this is not so if one replaces the radius by a suitable *pair* of arcs, or narrows the class of functions.

Let $\mathscr{E}$ be an at most enumerable set of arcs at unity (the point 1) and ζ be a point of K. Then by $\mathscr{E}_\zeta$ we mean the set of arcs at ζ obtained by rotating each arc in $\mathscr{E}$ about the origin through the angle arg ζ. We define $B_{\mathscr{E}}(f)$ to be the set of points $\zeta \in K$ to which there correspond two arcs at ζ on which f is bounded, such that these two arcs are separated by some arc in $\mathscr{E}_\zeta$.

THEOREM 2. *Let $f(z)$ be holomorphic in U. Suppose that the set $B_{\mathscr{E}}(f) \cap A$ is of second category. Then either $\Gamma_\rho(f, A)$ is of positive linear measure or f is identically constant.*

PROOF. A consequence of [6, p. 423, Theorem] is that for every point $\zeta \in B_{\mathscr{E}}(f) \cap A$ with the exception of a set of points of first category, $C_\rho(f, \zeta)$ is a proper subset of Ω. This means that $E(f) \cap A$ is of second category, and Theorem 2 now follows from Theorem C_2.

Let $\mathscr{E}$ be the set consisting of a single arc, the radius at unity. According to [3, p. 82, Corollary 1], there exists a function $f(z)$, meromorphic and not identically constant in U, such that at every $\zeta \in K$ there are two chords at ζ along which $f(z) \to 0$ as $z \to \zeta$, and these two chords are separated by the radius at ζ. This example shows that Theorem 2 does not hold for meromorphic functions.

By a nontangential arc at a point $\zeta \in K$ we mean an arc at ζ that lies in some Stolz angle with vertex ζ. Define $G(f)$ to be the set of points $\zeta \in K$ at which there exists a nontangential arc on which f is bounded.

THEOREM 3. *Let $f(z)$ be holomorphic and normal in U. Suppose that the set $G(f) \cap A$ is of second category. Then either $\Gamma_\rho(f, A)$ is of positive linear measure or f is identically constant.*

PROOF. It follows from [7, p. 403, Theorem 5] that for every point $\zeta \in G(f) \cap A$ with the exception of a set of points of first category, $C_\rho(f, \zeta)$ is a proper subset of Ω. The proof is now completed as was the proof of Theorem 2.

Theorem 1 shows that in Theorem 3, the condition that $f(z)$ be normal cannot be dropped.

Theorem 3 does not hold for meromorphic normal functions. For there exists (see the proofs of Theorems 3 and 6 in [7]) a Schwarzian triangle-function $f(z)$ such that at every point $\zeta \in K$ there are two nontangential arcs, separated by the radius at ζ, on which $f(z)$ is bounded, but f has no asymptotic value whatsoever.

References

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